

SUPERSPINORS

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ABSTRACT. We propose to replace $\text{Spin}(1,3)^c$ as the space-time symmetry group of quantum field theory by a compact semisimple Lie group. The results are rendered via the formalism of superspinors - objects identifiable as particle or antiparticle wave functions, and governed by the Fermi-Dirac statistics.

1. INTRODUCTION

In this paper we attempt to replace the Lorentz group of space-time symmetries with a compact semisimple Lie group of symmetries of purely quantum objects. The rationale behind such an attempt is deceptively simple: the quantum field theoretical data ought to transform unitarily. Admittedly, our approach is not the only one. There are unitary representations of the classical Lorentz group, and if the technical difficulties posed by their infinite dimensionality are overcome, there would appear to be no call for replacing the group. Be that as it may, there is another compelling reason to look for a different symmetry group. The seeming disparity in the way electrons and positrons are treated [5] has to be either explained in terms of fundamental space-time symmetries or done away with, and the present physical setup does not do that.

There were several attempts in the past. The most notable and fruitful replacement candidate had been the conformal group. Even though those transformations only leave the light cone intact, while wrecking havoc on the time-like dynamics, such achievements as conformal field theory, the Penrose transform and the formalism of twistors in the curved space-time [12], [13] had validated that particular break from the grip of the Lorentz group, as well as inspired further research.

Whatever the motivation, this replacement ushers in some new features, akin to supersymmetric theories, and has experimentally verifiable consequences. Instead of considering Dirac spinors Ψ and Φ delineating particles and antiparticles as separate entities, we unify them. Mathematically, this unification is expressed by the formalism of superspinors - objects appearing to different observers as particle or antiparticle wave functions, depending on a particular frame of reference (parameterized by the frame rapidity α):

$$\Psi(\alpha + \pi) = \pm \bar{\Phi}(\alpha).$$

To elicit these transformations, we no longer require the standard Dirac Lagrangian:

$$\mathcal{L}_D = \frac{i}{2}(\Psi^\dagger \gamma^\mu \partial_\mu \Psi - \partial_\mu \Psi^\dagger \gamma^\mu \Psi - m \Psi^\dagger \Psi).$$

Instead, we introduce a modified Lagrangian:

$$\mathcal{L}_D = \frac{i}{2}(\Psi^\dagger \gamma^\mu \nabla_\mu(\alpha) \Psi - \nabla_\mu(\alpha) \Psi^\dagger \gamma^\mu \Psi - m \Psi^\dagger \Psi),$$

where $\nabla_\mu(\alpha)$ is a family of principal connections subjected to the relativistic constraint

$$g^{\nu\eta}(\alpha)\nabla_\nu(\alpha)\nabla_\eta(\alpha) = \partial_\mu\partial^\mu,$$

and the metric $g^{\nu\eta}(\alpha)$ is induced by the deformations germane to the new group.

Another curious feature of the new group is that superspinors are fermions *par excellence*. That is to say, in the course of second quantization some appropriate anticommutators vanish as a result of fairly natural assumptions.

Novelties notwithstanding, the rotation properties of spinors go over to superspinors, owing to the $SU(2)$ subgroup common to both the Lorentz group and the new one. Therefore, superspinors in a rest frame coincide with the Dirac spinors.

A few words about the paper. Its organization is straightforward: first we develop the necessary Lie group theory in Sections 2 and 3, then to make it usable we modify the concept of free spin structure in Section 4, and finally Sections 5, 6, 7 contain some application of the aforementioned mathematics to the spinorial representations. Section 8 is not as rigorous, an homage to the experimental aspects of superspinors.

Lastly, we dispense with the physical constants by setting $\hbar = c = 1$.

2. COMPACTIFICATION OF THE SYMMETRY GROUP

The point of departure for a symmetry group search is the consideration of the oriented Grassmanian manifold \mathfrak{Gr}_6^{+3} of 3-planes in \mathbb{R}^6 as a natural arena to tackle the inertial frames of \mathbb{R}^4 . Just a glimpse of what we are up against. A boost in the x direction is given by

$$x' = \frac{x + vt}{\sqrt{1 - v^2}}, \quad t' = \frac{t + vx}{\sqrt{1 - v^2}}. \quad (2.1)$$

As $v \rightarrow 1$, we have

$$\lim_{v \rightarrow 1} \left(\arcsin \left(\frac{vt}{\sqrt{x^2 + v^2 t^2}} \right) - \arcsin \left(\frac{t}{\sqrt{t^2 + v^2 x^2}} \right) \right) = 0. \quad (2.2)$$

Not only are there infinite lengths (which can be easily normalized away), but also the frames as such cease to exist at $v = 1$. As demonstrated by (2.2), the x and t axes merge. That necessitates a representation of the Lorentz frames by points of some projective variety. Our choice (to be justified in due course) is \mathfrak{Gr}_6^{+3} .

Fortuitously, $\mathfrak{Gr}_6^{+3} \subset \mathbb{S}^{19}$ (the latter being the unit sphere in \mathbb{R}^{20}). This allows us to use the Plücker coordinates (and the quadratic Plücker relations since \mathfrak{Gr}_6^{+3} is a proper subvariety of \mathbb{S}^{19}). For a comprehensive reference on the Plücker coordinates, see the classic by Hodge and Pedoe ([8], Chapter VII). Thus given a 3×6 matrix of rank 3, there are precisely 20×3 minors (not counting the column permutations). Their determinants are not all zero because of the rank condition, and comprise the set of Plücker coordinates of the 3-plane spanned by the row vectors. These are unique up to a common positive multiple. We denote them by $p_{i_0 i_1 i_2}$, where i_λ 's are distinct numbers from the set $(0, 1, 2, 3, 4, 5)$ with the additional property $i_0 < i_1 < i_2$. We arrange the 20 $p_{i_0 i_1 i_2}$'s in lexicographic order. The aggregate of these entities can be thought of as a surjective mapping P from the set of all 3×6 matrices of rank 3 onto the Grassmanian.

A point $P \in \mathfrak{Gr}_6^{+3}$ is completely determined by its coordinates:

$$P = (p_{012}, p_{013}, p_{014}, \dots, p_{345}).$$

$$F_{i_0 i_1 j_0 j_1 j_2 j_3}(P) = 0, \quad (2.3)$$
$$F_{i_0 i_1 j_0 j_1 j_2 j_3}(P) = \sum_{\lambda=0}^3 (-1)^\lambda p_{i_0 i_1 j_\lambda} p_{j_0 \dots j_{\lambda-1} j_{\lambda+1} \dots j_3}. \quad (2.4)$$

We fix a point $P_0 \in \mathfrak{Gr}_6^3$, and choose the Plücker coordinates so that

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \quad (2.5)$$

$$\hat{\alpha}_k = \arctan \tanh \left(\frac{\frac{\partial x^k}{\partial t}}{\sqrt{1 - \left(\frac{\partial x^k}{\partial t} \right)^2}} \right), \quad \hat{\alpha}_k \in [0, \frac{\pi}{4}), \quad k = \{1, 2, 3\}, \quad (2.6)$$
$$\begin{bmatrix} \cos 4\widehat{\alpha}_1 & 0 & 0 & \sin 4\widehat{\alpha}_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\sin 4\widehat{\alpha}_1 & 0 & 0 & \cos 4\widehat{\alpha}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (2.7)$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos 4\widehat{\alpha}_2 & 0 & \sin 4\widehat{\alpha}_2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\sin 4\widehat{\alpha}_2 & 0 & \cos 4\widehat{\alpha}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (2.8)$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos 4\hat{\alpha}_3 & \sin 4\hat{\alpha}_3 & 0 & 0 \\ 0 & 0 & -\sin 4\hat{\alpha}_3 & \cos 4\hat{\alpha}_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.9)$$

followed by P . Thereafter we denote (2.7)-(2.9) by $\widehat{\alpha}_k$, and the image points of individual $P \circ \widehat{\alpha}_k$ by $P(\widehat{\alpha}_k)$.

If instead of (2.5) we choose some other matrix of rank 3, yielding the same Plücker coordinates, the latter would be related to (2.5) by a matrix $A \in GL(3, \mathbb{R})^e$ so that the span of row space would remain unchanged. Therefore (2.7)-(2.9) are independent of the choice of (2.5). Furthermore, the diagram below would commute ($\times|A|$ means scalar multiplication by $\det A$).

$$\begin{array}{ccc} \widehat{\alpha}_k & \xrightarrow{P} & P \circ \widehat{\alpha}_k \\ A \downarrow & & \downarrow \times|A| \\ A\widehat{\alpha}_k & \xrightarrow{P} & P \circ (A\widehat{\alpha}_k) \end{array}$$

Such a swap will always preserve the resulting Plücker coordinates.

At this point we state and prove an important theorem regarding the properties of (2.7)-(2.9).

Theorem 2.1. *The mappings*

$$P \circ \widehat{\alpha}_k : \mathbb{R}^+ \hookrightarrow \mathfrak{Gr}^+_{6,3}$$

effected by the composition of P and (2.7)-(2.9) are analytic embeddings.

Proof. We prove the theorem for a particular case ($k = 1$). The remaining cases would then follow *mutatis mutandis*. From (2.7) we construct the transformation matrix:

$$\begin{bmatrix} \cos 4\widehat{\alpha}_1 & 0 & 0 & \sin 4\widehat{\alpha}_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad (2.10)$$

The list of all nonzero (for some $\widehat{\alpha}_k$) Plücker coordinates is as follows:

$$\begin{aligned} p_{012} &= \cos 4\widehat{\alpha}_1, \\ p_{123} &= \sin 4\widehat{\alpha}_1. \end{aligned} \quad (2.11)$$

From this, the analyticity is immediate. Demonstrating the one-to-oneness is slightly more involved. For the time being we think of $P \circ \widehat{\alpha}_k$ as a mapping into \mathbb{S}^{19} . This is a 19-dimensional analytic manifold. On this manifold, there is a natural system of charts $(V_{p_{i_0 i_1 i_2}}, z_{m_1 m_2 m_3})$ such that $V_{p_{i_0 i_1 i_2}}$ is the family of hypersurfaces with $p_{i_0 i_1 i_2} \neq 0$, analytically diffeomorphic to an open subset of \mathbb{R}^{19} coordinatized by

$$z_{m_1 m_2 m_3} = \frac{p_{m_1 m_2 m_3}}{p_{i_0 i_1 i_2}}, \quad m_1 m_2 m_3 \neq \sigma(i_0)\sigma(i_1)\sigma(i_2) \quad \forall \sigma.$$

In our case, $p_{i_0 i_1 i_2} = p_{012}$ or $p_{i_0 i_1 i_2} = p_{123}$. The image is contained in $V_{p_{012}}$ and $V_{p_{123}}$. Hence the existence of a smooth nonvanishing tangent vector to our image curve would suffice. The respective non-zero components of the tangent vector are:

$$\begin{aligned} z'_{123} &= \frac{-4}{\sin^2 4\widehat{\alpha}_1} \quad \text{within } V_{p_{012}}, \\ z'_{012} &= \frac{4}{\cos^2 4\widehat{\alpha}_1} \quad \text{within } V_{p_{123}}. \end{aligned}$$

Their being smooth and nonvanishing clinches the proof. \square

We use the notation LGr_{\curvearrowright} to name the aggregate image of $P \circ \hat{\alpha}_k$, $k = \{1, 2, 3\}$. The geometric meaning of LGr_{\curvearrowright} is transparent. Each point represents a 3-plane transversal to the natural foliation of \mathbb{R}^4 by the hyperbolic hypersurfaces parameterized via $(t^2 - (x^{k_1})^2 = a > 0, x^{k_2}, x^{k_3})$, and intersecting every leaf of that foliation. All the planes generated by a boost in any particular direction assembled would reconstitute the inside of the corresponding light wedge $t^2 - (x^k)^2 = 0$. We are as yet to establish a link between our construct and the classical Lorentz group. Our claim is, there is such a link, and, in fact, the set of all spatially rotated boost frames $SO(3) \cdot LGr_{\curvearrowright}$ accounts for all orthochrone boosts. Let us take a look at the $SO(1, 3)^e/SO(3)$ bundle over $SO(3) \cdot LGr_{\curvearrowright}$. Define a canonical section of this bundle by

$$\mathcal{S} : P(O \cdot \hat{\alpha}_k) \longmapsto B(O \cdot \hat{\alpha}_k). \quad (2.12)$$

Here $B(\hat{\alpha}_k)$ are the standard 4×4 matrices representing the boosts of $SO(1, 3)^e$. This section is very nice. We have

Proposition 2.1. \mathcal{S} defined by (2.12),

$$\mathcal{S} : SO(3) \cdot LGr_{\curvearrowright} \longrightarrow SO(1, 3)^e/SO(3), \quad \text{is a diffeomorphism.}$$

Proof. This is just an elementary application of the Cartan's 'technique of the graph' ([3], for modern treatment see [6], Lecture 6). We show that the diagonal subset

$$\Delta = \{\hat{\alpha}_k \mid P(O \cdot \hat{\alpha}_k) \times B(O \cdot \hat{\alpha}_k)\}$$

projects diffeomorphically onto the base and the fiber. The system of charts used in the proof of Theorem 2.1, from the standard Euclidean metric, induces an analytic Riemannian metric on \mathbb{S}^{19} , ergo on \mathfrak{Gr}^{+3}_6 . Employing this metric and the tangent vectors obtained in the course of proving Theorem 2.1, we get global dual forms $d\hat{\alpha}_k$. Those are analytic and invariant with respect to the action of $P \circ \hat{\alpha}_k$. Next we take a right-invariant coframe Ω_j on $SO(1, 3)/SO(3)$. The exterior differential system

$$\Delta^* = \pi_{\text{base}}^* d\hat{\alpha} - \pi_{\text{fiber}}^* \Omega \quad (2.13)$$

is completely integrable and defines an analytic foliation of the diagonal subset. To see the injectivity of $\pi_{\text{base}*}$, we assume $\pi_{\text{base}*}(X) = 0$, for some vector field tangent to the foliation. Then

$$0 = i_X \Delta^* = i_X \pi_{\text{base}}^* (\omega, d\hat{\alpha}) - i_X \pi_{\text{fiber}}^* \Omega = -i_X \pi_{\text{fiber}}^* \Omega. \quad (2.14)$$

But Ω is a full coframe, hence $\pi_{\text{fiber}*}(X) = 0$, and $X = 0$. Since the dimensions of the foliation and the base are the same and $\pi_{\text{base}*}$ is injective, it is an isomorphism. Now we apply the Inverse Function Theorem to deduce that the restriction of $\pi_{\text{base}*}$ to every leaf is a local diffeomorphism that happens to be invariant under the group action on the right. Therefore it is a global diffeomorphism. In a similar vein we deal with $\pi_{\text{fiber}*}$. By uniqueness, Δ is the graph of \mathcal{S} , and the proposition now follows. \square

LGr_{\curvearrowright} is not closed in the quotient topology of \mathfrak{Gr}^{+3}_6 . Now we manufacture the set $\overline{LGr_{\curvearrowright}}$. To be able to adjoin the limiting points, we have to check if they are *bona fide* elements of the Grassmanian. This amounts to verifying these two relations: the easy one-

$$\lim_{\hat{\alpha}_k \rightarrow \frac{\pi}{4}} p_{\sigma(i_0)\sigma(i_1)\sigma(i_2)} = \text{sign } \sigma \lim_{\hat{\alpha}_k \rightarrow \frac{\pi}{4}} p_{i_0 i_1 i_2}, \quad (2.15)$$

and the cumbersome one - (2.4);

$$\lim_{\widehat{\alpha}_k \rightarrow \frac{\pi}{4}} F_{i_0 i_1 j_0 j_1 j_2 j_3}(P(\widehat{\alpha}_k)) = 0. \quad (2.16)$$

Luckily for us, due to the paucity of nonzero $p_{i_0 i_1 i_2}$'s, there is only one nontrivial identity (disregarding index permutations) for each transformation (2.7)-(2.9). We have

$$\begin{aligned} \lim_{\widehat{\alpha}_1 \rightarrow \frac{\pi}{4}} F_{120123}(P(\widehat{\alpha}_1)) &= \lim_{\widehat{\alpha}_1 \rightarrow \frac{\pi}{4}} (p_{012}p_{123} - p_{123}p_{012}) \\ &= \lim_{\widehat{\alpha}_1 \rightarrow \frac{\pi}{4}} (\cos 4\widehat{\alpha}_1 \sin 4\widehat{\alpha}_1 - \sin 4\widehat{\alpha}_1 \cos 4\widehat{\alpha}_1) \\ &= 0 \quad \text{for (2.7),} \end{aligned} \quad (2.17)$$

$$\begin{aligned} \lim_{\widehat{\alpha}_2 \rightarrow \frac{\pi}{4}} F_{020123}(P(\widehat{\alpha}_2)) &= \lim_{\widehat{\alpha}_2 \rightarrow \frac{\pi}{4}} (p_{012}p_{023} - p_{023}p_{012}) \\ &= \lim_{\widehat{\alpha}_2 \rightarrow \frac{\pi}{4}} (\cos 4\widehat{\alpha}_2 (-\sin 4\widehat{\alpha}_2) - (-\sin 4\widehat{\alpha}_2) \cos 4\widehat{\alpha}_2) \\ &= 0 \quad \text{for (2.8),} \end{aligned} \quad (2.18)$$

$$\begin{aligned} \lim_{\widehat{\alpha}_3 \rightarrow \frac{\pi}{4}} F_{010123}(P(\widehat{\alpha}_3)) &= \lim_{\widehat{\alpha}_3 \rightarrow \frac{\pi}{4}} (p_{012}p_{013} - p_{013}p_{012}) \\ &= \lim_{\widehat{\alpha}_3 \rightarrow \frac{\pi}{4}} (\cos 4\widehat{\alpha}_3 \sin 4\widehat{\alpha}_3 - \sin 4\widehat{\alpha}_3 \cos 4\widehat{\alpha}_3) \\ &= 0 \quad \text{for (2.9).} \end{aligned} \quad (2.19)$$

For all three transformations, there is just one limiting point:

$$\lim_{\widehat{\alpha}_k \rightarrow \frac{\pi}{4}} P(\widehat{\alpha}_k) = P_\infty = (-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

We have succeeded in building $\overline{LGr}_\infty = LGr_\infty \cup P_\infty$. But a larger question is still looming: how to complete \overline{LGr}_∞ to a group space for some one-parameter subgroups of $SO(6)$? Evidently we need more Lorentz boost frames. The problem is, according to Proposition 2.1, all those frames are represented by the points of LGr_∞ . The frames in the other component of $SO(1,3)$ are essentially parachrone boosts and cannot be connected to $SO(1,3)^e$ (or any representation of it) via continuous transformations. The same is true of the remaining two components of the classical Lorentz group. To assuage this deficiency, we introduce the notion of 'virtual frame'. Within our realm we represent frames by the points of \mathfrak{Gr}_6^{+3} , i. e. by the appropriately positioned 3-planes in \mathbb{R}^6 . Now to complete \overline{LGr}_∞ we use the symmetry properties of \mathfrak{Gr}_6^{+3} . In keeping with the physical world, we supply a more concrete description. Thus, the 'virtual frames' correspond to the situation wherein 3-planes assembled would reconstitute the outside of the light wedge $t^2 - (x^k)^2 = 0$. Going from a 'real frame' to a 'virtual frame' amounts to flipping the signature of the Lorentzian metric involved. From our viewpoint, the virtuality is manifested in the parameters being reciprocal to those of (2.7)-(2.9):

$$\widehat{\alpha}_k = \arctan \tanh \left(\frac{\frac{\partial t}{\partial x^k}}{\sqrt{1 - \left(\frac{\partial t}{\partial x^k}\right)^2}} \right), \quad \widehat{\alpha}_k \in [0, \frac{\pi}{4}), \quad k = \{1, 2, 3\}. \quad (2.20)$$

The corresponding transformations are listed below:

$$\begin{bmatrix} -\cos 4\hat{\alpha}_1 & 0 & 0 & -\sin 4\hat{\alpha}_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \sin 4\hat{\alpha}_1 & 0 & 0 & -\cos 4\hat{\alpha}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (2.7')$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\cos 4\hat{\alpha}_2 & 0 & -\sin 4\hat{\alpha}_2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \sin 4\hat{\alpha}_2 & 0 & -\cos 4\hat{\alpha}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (2.8')$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\cos 4\hat{\alpha}_3 & -\sin 4\hat{\alpha}_3 & 0 & 0 \\ 0 & 0 & \sin 4\hat{\alpha}_3 & -\cos 4\hat{\alpha}_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (2.9')$$

Just as before, the above transformations possess the expected properties.

Theorem 2.2. *The mappings*

$$\mathbf{P} \circ \hat{\alpha}_k : \mathbb{R}^+ \hookrightarrow \mathfrak{Gr}^+_{6^3}$$

effected by the composition of \mathbf{P} and (2.7')-(2.9') are analytic embeddings.

The limiting process works as well:

$$\lim_{\hat{\alpha}_k \rightarrow \frac{\pi}{4}} P(\hat{\alpha}_k) = P_0 = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

The sets LGr_{\curvearrowright} and $\overline{LGr_{\curvearrowright}}$ are not connected so that the common boundary set is

$$\overline{LGr_{\curvearrowright}} \cap \overline{LGr_{\curvearrowright}} = \{P_0, P_{\infty}\}. \quad (2.21)$$

Now we act on $\overline{LGr_{\curvearrowright}}$ by spatial rotations. The resulting augmented set is

$$(SO(3) \cdot \overline{LGr_{\curvearrowright}}) \subset \mathfrak{Gr}^+_{6^3}. \quad (2.22)$$

Similarly, we obtain $(SO(3) \cdot \overline{LGr_{\curvearrowright}}) \subset \mathfrak{Gr}^+_{6^3}$. An essential relation holding true for those sets is that

$$(SO(3) \cdot \overline{LGr_{\curvearrowright}}) \cap (SO(3) \cdot \overline{LGr_{\curvearrowright}}) = \{P_0, P_{\infty}\}. \quad (2.23)$$

As it turns out, $\mathfrak{Gr}^+_{6^3}$ is the lowest-dimensional projective space with enough room to accommodate (2.23). That is possible only if

$$\dim \mathfrak{Gr}^+_{m^l} = \binom{m}{l} - 1 \geq 12. \quad (2.24)$$

From this one readily sees that $m \geq 6$. By contrast, the common boundary of $(SO(3) \cdot \overline{LGr_{\curvearrowright}})$ and $(SO(3) \cdot \overline{LGr_{\curvearrowright}})$ embedded in $\mathfrak{Gr}^+_{4^3}$ would have been homeomorphic to \mathbb{S}^2 , and because of it being connected there would be ways to move

from P_0 to P_∞ via spatial rotations.

An important consequence of Theorem 2.1, Theorem 2.2, and (2.21) is the following statement:

Theorem 2.3.

$$H^1(\overline{LGr_\curvearrowright} \cup \overline{LGr_\curvearrowleft}, \mathbb{Z}) \cong \mathbb{Z}^{15}.$$

Now we are in a position to unveil the Lie algebra $\widehat{\mathfrak{g}} \cong \mathfrak{so}(1, 3)$ underpinning (2.7)-(2.9) and (2.7')-(2.9') (which from this point on are parametrized by α_k). To begin with, we express the boosts in terms of the standard orthogonal Lie algebra basis:

$$\widehat{K}_1 = \begin{bmatrix} 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (2.25)$$

$$\widehat{K}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (2.26)$$

$$\widehat{K}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (2.27)$$

Their brackets

$$[\widehat{K}_1, \widehat{K}_2] = i\widehat{J}_3, \quad [\widehat{K}_2, \widehat{K}_3] = i\widehat{J}_1, \quad [\widehat{K}_3, \widehat{K}_1] = i\widehat{J}_2, \quad (2.28)$$

yield the rotations:

$$\widehat{J}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (2.29)$$

$$\widehat{J}_2 = \begin{bmatrix} 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (2.30)$$

$$\widehat{J}_3 = \begin{bmatrix} 0 & i & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (2.31)$$

The algebra generated by \widehat{K} 's and \widehat{J} 's is closed:

$$\begin{aligned} [\widehat{J}_1, \widehat{J}_2] &= i\widehat{J}_3, & [\widehat{J}_2, \widehat{J}_3] &= i\widehat{J}_1, & [\widehat{J}_3, \widehat{J}_1] &= i\widehat{J}_2, \\ [\widehat{J}_1, \widehat{K}_2] &= i\widehat{K}_3, & [\widehat{J}_1, \widehat{K}_3] &= -i\widehat{K}_2, & [\widehat{J}_2, \widehat{K}_1] &= i\widehat{K}_3, \\ [\widehat{J}_2, \widehat{K}_3] &= i\widehat{K}_1, & [\widehat{J}_3, \widehat{K}_1] &= i\widehat{K}_2, & [\widehat{J}_3, \widehat{K}_2] &= -i\widehat{K}_1, \end{aligned} \quad (2.32)$$

and all the remaining brackets vanish.

The upshot of our discourse is that compactification must involve the adjoining of virtual frames. Indeed, the parameters of the classical Lorentz group run through the set of nonnegative real numbers; this set is not bounded, therefore no point identification or creation of a compact group space is possible prior to taking some kind of closure. But once the virtual frames are in, we are forced to treat them just as we would the inertial frames. In particular, an observer situated inside would have no means to decide whether their frame is real or virtual. Consequently, all the foregoing constructing may start off with the virtual frames as a foundation. That way one obtains an alternative algebra denoted $\widehat{\mathfrak{g}}$ instead of $\widehat{\mathfrak{g}}$. The two are isomorphic but nonetheless not identical. We have

$$\widehat{K}_3 = \widehat{K}_3, \quad (2.33)$$

$$\widehat{K}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (2.34)$$

$$\widehat{K}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \end{bmatrix}. \quad (2.35)$$

Proceeding as before, we build $\widehat{\mathfrak{g}} \subset \mathfrak{so}(6)$, such that $\widehat{\mathfrak{g}} \cong \mathfrak{so}(1,3)$. $\widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{g}}$ are isomorphic, and $[\widehat{J}_i, \widehat{J}_j] = 0$ for all pairs (i, j) . Those commutative brackets enable us to define a new entity - one that is completely invariant with respect to the vantage point change - $\widehat{\mathfrak{g}} \rtimes \widehat{\mathfrak{g}} \subset \mathfrak{so}(6)$. Our joint algebra is a closed subalgebra of $\mathfrak{so}(6)$, $\dim \widehat{\mathfrak{g}} \rtimes \widehat{\mathfrak{g}} = 6$. Its elements are generated by $J_i = \widehat{J}_i + \widehat{J}_i$ and $K_3 = \widehat{K}_3 = \widehat{K}_3$. In view of $\widehat{\mathfrak{g}} \cap \widehat{\mathfrak{g}} = \mathbb{R}K_3$, $\widehat{\mathfrak{g}} \rtimes \widehat{\mathfrak{g}} \neq \widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}$ - a nuance figuring

prominently in the following sections. According to Helgason ([7], Chapter II, §2, Theorem 2.1), there is a unique connected Lie subgroup of $SO(6)$, whose Lie algebra is the subalgebra $\widehat{\mathfrak{g}} \rtimes \widehat{\mathfrak{g}}$ of $\mathfrak{so}(6)$. Furthermore, by a fundamental result of Mostow [10], any semisimple Lie subgroup H of a compact Lie group C is closed in the relative topology of C . In our case, $SO(6)$ is compact, $\widehat{\mathfrak{g}} \rtimes \widehat{\mathfrak{g}}$ is semisimple, so that the Mostow's theorem applies. Thus we finally obtain

$$G \stackrel{\text{def}}{=} \{\exp iX \mid X \in \widehat{\mathfrak{g}} \rtimes \widehat{\mathfrak{g}}\}. \quad (2.36)$$

The group herein defined by (2.36) ought to replace the classical Lorentz group as the symmetry group of quantum objects - the only objects surmised to be capable of being virtual.

3. UNITARY CONVERSION

Having thus determined the structure of the group we begin to look for an appropriate spinor representation of G . We utilize a well-known isomorphism of Lie algebras. Specifically, $\mathfrak{so}(6) \cong \mathfrak{su}(4)$. Via the above isomorphism, we find a closed subalgebra $\widehat{\mathfrak{u}}\mathfrak{g} \rtimes \widehat{\mathfrak{u}}\mathfrak{g} \subset \mathfrak{su}(4)$,

$$\widehat{\mathfrak{u}}\mathfrak{g} \rtimes \widehat{\mathfrak{u}}\mathfrak{g} \cong \widehat{\mathfrak{g}} \rtimes \widehat{\mathfrak{g}}. \quad (3.1)$$

Once more invoking ([7], Chapter II, §2, Theorem 2.1), and [10], we get a closed subgroup of $SU(4)$. This subgroup, denoted UG , is the unitary counterpart of G . The basis of $\widehat{\mathfrak{u}}\mathfrak{g} \rtimes \widehat{\mathfrak{u}}\mathfrak{g}$ (with the notation retained from Section 2) is

$$\begin{aligned} J_1 &= \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, & J_2 &= \frac{1}{2} \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{bmatrix}, \\ J_3 &= \frac{1}{2} \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}, & K_3 &= \frac{1}{2} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \\ K_2 &= \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, & K_1 &= \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \quad (3.2)$$

$\widehat{\mathfrak{u}}\mathfrak{g} \rtimes \widehat{\mathfrak{u}}\mathfrak{g}$ decomposes as a vector space into two three-dimensional subspaces,

$$\widehat{\mathfrak{u}}\mathfrak{g} \rtimes \widehat{\mathfrak{u}}\mathfrak{g} = \mathfrak{j} \oplus \mathfrak{k}, \quad (3.3)$$

of which \mathfrak{j} is a closed compactly embedded Lie subalgebra generated by J_i , and the following identities hold:

$$[\mathfrak{j}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{j}. \quad (3.4)$$

The corresponding group

$$SU_J(2) \stackrel{\text{def}}{=} \{\exp iJ \mid J \in \mathfrak{j} \subset \widehat{\mathfrak{u}}\mathfrak{g} \rtimes \widehat{\mathfrak{u}}\mathfrak{g}\} \quad (3.5)$$

is a closed subgroup of UG .

The most significant property of UG is that it serves as a covering space for G . Formally we have

Theorem 3.1. *There exists a map Ξ , such that*

$$\Xi : \quad UG \longrightarrow G$$

is a twofold covering epimorphism of Lie groups.

Proof. We know that the group $SU_J(2)$ is a twofold cover for $SO(3) \subset G$. That covering property may be expressed by

$$\left. \begin{matrix} I_{UG} \\ -I_{UG} \end{matrix} \right\} \xrightarrow{\Xi_{SU_J(2)}} I_G.$$

From (3.1) and ([7], Chapter II, §1, Theorem 1.11) we extract a local isomorphism between UG and G . That means there are sets O_{UG}^+ and O_{UG}^- , open in UG , $I_{UG} \in O_{UG}^+$, $-I_{UG} \in O_{UG}^-$, satisfying $O_{UG}^+ \cap O_{UG}^- = \emptyset$, and local diffeomorphisms f^+ and f^- , such that $f^+(O_{UG}^+) = f^-(O_{UG}^-) = O_G$, the latter set being an open neighborhood of identity in G . Shrinking O_{UG}^+ if necessary, we find an open neighborhood of identity in UG which we call O , $O \subset O_{UG}^+$ that is particularly amenable to the group multiplication on the left. Namely, $(-I_{UG} \cdot O) \subset O_{UG}^-$. Because of the group structure, we have

$$(u \cdot O) \cap (u \cdot (-I_{UG} \cdot O)) = \emptyset, \quad \forall u \in UG.$$

Now UG is compact and has no small subgroups, that is, given an open set $T \subset UG$, such that the diameter of T with respect to the natural left-invariant Killing metric, $\text{diam}(T) < \text{diam}(O)$, there exists an element $u_T \in O$, and a nonnegative integer n_T with the property $T \subset (u_T^{n_T} \cdot O)$.

We define Ξ by extending f^+ :

$$\Xi(u_T^{n_T} \cdot O) \stackrel{\text{def}}{=} (f^+(u_T))^{n_T} f^+(O).$$

The group operation on the right is differentiable, in fact analytic, so Ξ is differentiable. We claim that Ξ is a twofold covering map. If $T \subset O_{UG}^-$, there are u_T , n_T such that $u_T^{n_T} = -I_{UG}$, $(f^+(u_T))^{n_T} = I_G$, so that $\Xi(T) \subset f^+(O)$, and G is evenly covered for any particular set of u_T 's and n_T 's. \square

In the sequel we will work with the homogeneous space $UG/SU_J(2)$. Its topology turns out to be crucial in our efforts to put UG on solid ground.

Theorem 3.2.

$$\pi_1(UG/SU_J(2)) = 0.$$

Proof. For all Lie groups $\pi_2(\cdot) = 0$ [2]; for $SU_J(2)$, $\pi_0(SU_J(2)) = 0$ by connectedness. Also, $SU_J(2)$ is a closed subgroup of $SU(4)$ in the ordinary matrix topology. We therefore have the following exact homotopy sequence [2]:

$$0 \rightarrow \pi_2(SU(4)/SU_J(2)) \rightarrow \pi_1(SU_J(2)) \rightarrow \pi_1(SU(4)) \rightarrow \pi_1(SU(4)/SU_J(2)) \rightarrow 0.$$

$\pi_1(SU(4)) = 0$ [2] whence

$$\pi_1(SU(4)/SU_J(2)) \cong \pi_1(SU_J(2)) = \pi_1(\mathbb{S}^3) = 0.$$

Now homotopy is functorial. The embedding $\xi : UG/SU_J(2) \hookrightarrow SU(4)/SU_J(2)$ induces the monomorphism of fundamental groups

$$\xi_{\pi*} : \pi_1(UG/SU_J(2)) \rightarrow \pi_1(SU(4)/SU_J(2)). \quad \square$$

Theorem 3.3.

$$UG/SU_J(2) \cong \mathbb{S}^3.$$

Proof. Based on the decomposition (3.3), there is an involutive automorphism

$$\vartheta : \widehat{\mathfrak{ug}} \rtimes \widehat{\mathfrak{ug}} \longrightarrow \widehat{\mathfrak{ug}} \rtimes \widehat{\mathfrak{ug}},$$

defined by

$$\vartheta(J + K) = J - K, \quad \forall J \in \mathfrak{j}, \quad \forall K \in \mathfrak{k}.$$

\mathfrak{j} is the set of fixed points of ϑ . It is unique ([7], Chapter IV, §3, Proposition 3.5).

The pair $(\widehat{\mathfrak{ug}} \rtimes \widehat{\mathfrak{ug}}, \vartheta)$ is an orthogonal symmetric Lie algebra ([7], Chapter IV, §3). There is a Riemannian symmetric pair $(UG, SU_J(2))$ associated with $(\widehat{\mathfrak{ug}} \rtimes \widehat{\mathfrak{ug}}, \vartheta)$ so that the quotient $UG/SU_J(2)$ is a complete locally symmetric Riemannian space. Furthermore, its curvature corresponding to any UG -invariant Riemannian structure is given by ([7], Chapter IV, §4, Theorem 4.2):

$$R(K_{i_1}, K_{i_2})K_{i_3} = -[[K_{i_1}, K_{i_2}], K_{i_3}] \quad \forall K_{i_1}, K_{i_2}, K_{i_3} \in \mathfrak{k}.$$

Computing the sectional curvature we see that $R^{\text{sect}} \equiv 1$. Now a pedestrian version of the Sphere theorem [4] asseverates that a complete simply connected Riemannian manifold with $R^{\text{sect}} \equiv 1$ is isometric to a sphere of appropriate dimension. In our case the topological condition is satisfied in view of Theorem 3.2. \square

4. SUPERSPIN STRUCTURES

The task ahead is clear: to find the relativistic transformation law for Dirac spinors. Any new group of symmetries (including the newly-minted G of Section 2) would still have to provide a bijective correspondence between two sets of solutions of the Dirac equation - one being the set of original spinors, the other being the set of transformed ones. At the same time this correspondence should not mess up the spatial rotations of spinors. Last, but not least, the resulting representation of G has to be irreducible to ensure there is no mass splitting [11].

To gain a better insight into the problem, prior to delving into the mire of formulas, we discuss the concept of ‘free spin structure’, originally proposed by Plymen and Westbury [14]. This discussion might guide us towards a reasonable definition of the superspin structure. Thus let M be a 4-dimensional smooth manifold with all the obstructions to the existence of a Lorentzian metric vanishing (for instance, a parallelizable M would do). Let

$$\Lambda : \text{Spin}(1, 3)^e \rightarrow SO(1, 3)^e$$

be the twofold covering epimorphism of Lie groups. A free spin structure on M consists of a principal bundle $\zeta : \Sigma \rightarrow M$ with structure group $\text{Spin}(1, 3)^e$ and a bundle map $\tilde{\Lambda} : \Sigma \rightarrow \mathcal{FM}$ into the bundle of linear frames for TM , such that

$$\tilde{\Lambda} \circ \tilde{R}_S = \tilde{R}'_{\iota \circ \Lambda(S)} \circ \tilde{\Lambda} \quad \forall S \in \text{Spin}(1, 3)^e,$$

$$\zeta' \circ \tilde{\Lambda} = \zeta,$$

\tilde{R} and \tilde{R}' being the canonical right actions on Σ and \mathcal{FM} respectively, $\iota : SO(1, 3)^e \rightarrow GL(4, \mathbb{R})$ the natural inclusion of Lie groups, and $\pi' : \mathcal{FM} \rightarrow M$ the canonical projection. The map $\tilde{\Lambda}$ is called a spin-frame on $\text{Spin}(1, 3)^e$. This definition of a spin structure induces metrics on Σ . Indeed, given a spin-frame $\tilde{\Lambda} : \Sigma \rightarrow \mathcal{FM}$, a dynamic metric $g_{\tilde{\Lambda}}$ is defined to be the metric that ensures orthonormality of all frames in $\tilde{\Lambda}(\Sigma) \subset \mathcal{FM}$. It should be emphasized that within the Plymen and Westbury’s formalism the metrics are built *a posteriori*, after a spin-frame has been set by the field equations.

There is no way to extrapolate the above definition onto our framework because our group in its present incarnation does not act on any 4-dimensional manifold. However, their idea of carving a metric out of the spin structure permits a not-so-literal generalization. A principal connection on the UG bundle over the physical space-time would quantify the amount by which a frame deviates from the standard Lorentz frame. Then there is a metric that compensates for the deviation in such a manner as to appear to an observer dwelling in that frame to be the standard Lorentz metric. To preserve the commutation relations among the impulse operators we must insist on the metric being flat. This, in turn, mandates the following extension of the Einstein's Equivalence Principle: locally every noninertial frame is equivalent to a metric. There are familiar rotating or accelerating frames, entailing curved metrics. We postulate, that, in addition to those frames, some purely quantum noninertial frames are equivalent to flat but nonetheless nonstandard metrics. Unlike rotating and accelerating frames however, the G -frames are globally equivalent to some nonstandard flat metrics.

We cannot eschew the representation of G on $\text{Diff}(\mathbb{R}^4)$. The presence of virtual frames effectively kills any chance of representing the group solely by inertial frames.

Now we set out to demonstrate that our program, spelled out above, is viable. Consider the natural inclusions of Lie groups

$$\iota : UG \hookrightarrow GL(4, \mathbb{C}), \quad \iota : \text{Spin}(1, 3)^e \hookrightarrow GL(4, \mathbb{C}). \quad (4.1)$$

Their images inside $GL(4, \mathbb{C})$ intersect:

$$\iota(UG) \cap \iota(\text{Spin}(1, 3)^e) = SU_J(2). \quad (4.2)$$

Because of (4.2), the set

$$\text{Ad}_{\iota(UG)}(\iota(\text{Spin}(1, 3)^e)) = \coprod_{u \in UG} u\text{Spin}(1, 3)^e u^H, \quad (4.3)$$

the disjoint union of conjugates of $\text{Spin}(1, 3)^e$, has the same cardinality as the set of all boosts in UG . Similarly, there is the natural inclusion

$$\iota : SO(4) \hookrightarrow GL(4, \mathbb{R}). \quad (4.4)$$

The set $\text{Ad}_{\iota(SO(4))}(\iota(SO(1, 3)^e))$ is homeomorphic to $SO(4)/SO(3) \cong \mathbb{S}^3$. Combining this with Theorem 3.3 we arrive at two strings of relations running parallel:

$$\begin{array}{ccccc} \text{Ad}_{\iota(UG)}(\iota(\text{Spin}(1, 3)^e)) & \xlongequal{\quad} & UG/SU_J(2) & \xrightarrow{\cong} & \mathbb{S}^3 \\ & & & \parallel & \\ \text{Ad}_{\iota(SO(4))}(\iota(SO(1, 3)^e)) & \xlongequal{\quad} & SO(4)/SO(3) & \xrightarrow{\cong} & \mathbb{S}^3 \end{array}$$

The double horizontal lines indicate set-theoretic bijective correspondences, the upper \cong is an isometry, the lower one is a diffeomorphism. Furthermore, the diagram below commutes and *de facto* defines the superspin structure as conjugation of the

free spin structure by the elements of UG .

$$\begin{array}{ccc}
\vec{\partial} & \xrightarrow{UG\text{-connection}} & E\vec{\partial} + K \\
\downarrow & & \downarrow \\
\text{Spin}(1, 3)^e & \xrightarrow{\text{Ad}_{\iota(UG)}} & e^{iK} \text{Spin}(1, 3)^e e^{-iK} \\
\downarrow & & \downarrow \\
\tilde{\Lambda} & \xrightarrow{\text{Ad}_{\iota(SO(4))}} & o\tilde{\Lambda}o^T \\
\downarrow & & \downarrow \\
g_{\tilde{\Lambda}} & \longrightarrow & og_{\tilde{\Lambda}}o^T
\end{array}$$

Thus the superspin structure is a way to link groups UG -conjugate to $\text{Spin}(1, 3)^e$ inside $GL(4, \mathbb{C})$, with those $SO(4)$ -conjugate to $SO(1, 3)^e$ inside $GL(4, \mathbb{R})$. All conceivable superspin structures are parameterized by the elements of $\text{Diff}(\mathbb{S}^3)$. In particular, they can be bunched together into equivalence classes parameterized by $\pi_3(\mathbb{S}^3) = \mathbb{Z}$.

The reason our definition has some nontrivial content is, the group $\text{Spin}(1, 3)^e$ features two inequivalent representations of $SO(1, 3)^e$ - $(1/2, 0)$ and $(0, 1/2)$. Had there been two equivalent ones, the set $\text{Ad}_{\iota(UG)}(\iota(\dots))$ would have consisted of only one element and the superspin structures would have been reduced to the free spin structures.

5. RELATIVISTIC COINVARIANCE

With the superspin structure in place we now nail down the particulars. Instead of the standard quantum field theory substitution

$$p_\mu \longrightarrow i\partial_\mu, \quad (5.1)$$

we employ the rule

$$p_\mu \longrightarrow i\nabla_\mu(\alpha) \stackrel{\text{def}}{=} i(\varepsilon_\mu^\nu(\alpha)\partial_\nu + i\kappa_\mu^a(\alpha)K_a), \quad (5.2)$$

$K_a \in \mathfrak{k}$, $\kappa_\mu^a(\alpha)$ being a superspinor potential, chosen to make $i\nabla_\mu(\alpha)$ a purely imaginary operator. $\nabla_\mu(\alpha)$ qualifies as a UG -connection on the principal UG -bundle over the physical space-time. Possibly, $\kappa_\mu^a(\alpha)$'s are functions of the base space coordinates. The case of the flat space-time can be elaborated at this point. Assuming flatness, $\kappa_{\mu_l}^a(\alpha)$ may depend only on x^0 and x^{μ_l} to properly convey the essence of the boost. Therefore, for a pure boost, only two of four $\kappa_\mu^a(\alpha)$'s are nonzero for a fixed a ; of those, one is $\kappa_0^a(\alpha)$. An additional restriction is entailed if we insist upon the Schrödinger representation being valid: $[p_\mu, p_\nu] = 0$. To that end we need

$$(\varepsilon_0^0(\alpha)\partial_0\kappa_{\mu_l}^a(\alpha) + \varepsilon_0^{\mu_l}(\alpha)\partial_{\mu_l}\kappa_{\mu_l}^a(\alpha)) - (\varepsilon_{\mu_l}^0(\alpha)\partial_0\kappa_0^a(\alpha) + \varepsilon_{\mu_l}^{\mu_l}(\alpha)\partial_{\mu_l}\kappa_0^a(\alpha)) = 0, \quad (5.3)$$

$$[\kappa_0^1(\alpha)K_1 + \kappa_0^2(\alpha)K_2 + \kappa_0^3(\alpha)K_3, \kappa_{\mu_l}^1(\alpha)K_1 + \kappa_{\mu_l}^2(\alpha)K_2 + \kappa_{\mu_l}^3(\alpha)K_3] = 0. \quad (5.4)$$

Now by virtue of $[\mathfrak{j}, \mathfrak{k}] = \mathfrak{k}$, for every space direction the corresponding boost must be obtainable via some $SU_J(2)$ action on (5.2). That action ought to be linear to

be truly spinorial:

$$\begin{aligned} U\gamma^\mu\nabla_\mu U^H &= U\gamma^\mu U^H \varepsilon_\mu^\nu \partial_\nu + i\kappa_\mu^a U\gamma^\mu U^H U K_a U^H \\ &= M_\eta^\mu \gamma^\eta \varepsilon_\mu^\nu \partial_\nu + M_\eta^\mu \gamma^\eta i\kappa_\mu^a r_a^n K_n \quad \text{by } [\mathbf{j}, \mathbf{k}] = \mathbf{k}. \end{aligned} \quad (5.5)$$

Here M_η^μ 's realize an $SO(3)$ transformation ($U \in SU_J(2)$), which is at its most transparent if γ^0 is diagonal. As for r_a^n 's, they determine how the potentials behave:

$$\tilde{\kappa}_\mu^a = \kappa_\mu^1 r_1^a + \kappa_\mu^2 r_2^a + \kappa_\mu^3 r_3^a, \quad \text{and} \quad (5.6)$$

$$|r_1^a|^2 + |r_2^a|^2 + |r_3^a|^2 = 1, \quad a = \{1, 2, 3\}. \quad (5.7)$$

In order for us to express $\kappa_\mu^a(\alpha)$ explicitly as functions of α , we have to introduce the concept of relativistic coinvariance. We define the relativistic coinvariance to be a twofold property of our mathematical formalism; that the impulse operators transform via a principal UG connection, and, at the same time, this connection complies with the relativistic invariance law

$$\boxed{\tilde{p}^\mu \tilde{p}_\mu = g^{\nu\lambda}(\alpha) \nabla_\nu(\alpha) \nabla_\lambda(\alpha) \stackrel{\text{def}}{=} g^{\nu\lambda}(0) \partial_\nu \partial_\lambda = p^\mu p_\mu} \quad (5.8)$$

translating to some algebraic relations between κ_μ 's. A pure boost is best exemplified by the boost in the x^{μ_l} -direction. The metric deforms as follows:

$$g^{00} = \cos \alpha, \quad g^{\mu_l \mu_l} = -\cos \alpha, \quad g^{0\mu_l} = g^{\mu_l 0} = \sin \alpha. \quad (5.9)$$

For that particular transform we have

$$(\kappa_0^2 - \kappa_{\mu_l}^2) \cos \alpha + 2\kappa_0 \kappa_{\mu_l} \sin \alpha = 0, \quad (5.10)$$

$$(\varepsilon_0^2 - \varepsilon_{\mu_l}^2) \cos \alpha + 2\varepsilon_0^0 \varepsilon_{\mu_l}^0 \sin \alpha = 1, \quad (5.11)$$

$$(\varepsilon_0^{\mu_l^2} - \varepsilon_{\mu_l}^{\mu_l^2}) \cos \alpha + 2\varepsilon_0^{\mu_l} \varepsilon_{\mu_l}^{\mu_l} \sin \alpha = -1, \quad (5.12)$$

$$(\varepsilon_0^0 \varepsilon_0^{\mu_l} - \varepsilon_{\mu_l}^0 \varepsilon_{\mu_l}^{\mu_l}) \cos \alpha + (\varepsilon_0^0 \varepsilon_{\mu_l}^{\mu_l} + \varepsilon_0^{\mu_l} \varepsilon_{\mu_l}^0) \sin \alpha = 0. \quad (5.13)$$

Without the no torsion assumption (which may be extraneous in the curved space-time), the last equation splits into

$$\begin{cases} (\varepsilon_0^0 \varepsilon_0^{\mu_l} - \varepsilon_{\mu_l}^0 \varepsilon_{\mu_l}^{\mu_l}) \cos \alpha = 0, \\ (\varepsilon_0^0 \varepsilon_{\mu_l}^{\mu_l} + \varepsilon_0^{\mu_l} \varepsilon_{\mu_l}^0) \sin \alpha = 0. \end{cases} \quad (5.14)$$

These boosts are not linear, generally speaking, yet with all the above-listed constraints the remaining arbitrariness is considerably less than the arbitrariness of internal symmetries and gauge transformations. It is reflected in the Lagrangian being given by a familiar expression [9]:

$$\mathcal{L}_D = \frac{i}{2} (\Psi^\dagger \gamma^\mu \nabla_\mu \Psi - \nabla_\mu \Psi^\dagger \gamma^\mu \Psi - m \Psi^\dagger \Psi). \quad (5.15)$$

The one crucial distinction we want to make is that in the present context, ∇_μ 's stand for components of a principal connection, rather than the metric connection.

If instead of (5.1) the minimal substitution

$$p_\mu - e\mathbb{A}_\mu \longrightarrow i\partial_\mu - e\mathbb{A}_\mu \quad (5.16)$$

is used, we set

$$p_\mu - e\mathbb{A}_\mu \longrightarrow i\nabla_\mu(\alpha) - e\mathbb{A}_\mu(\alpha) \stackrel{\text{def}}{=} i(\varepsilon_\mu^\nu(\alpha) \partial_\nu + \kappa_\mu^a(\alpha) K_a) - e\mathbb{A}_\mu(\alpha). \quad (5.17)$$

The coinvariance condition (5.8) then becomes

$$\begin{aligned} (\tilde{p}^\mu - e\tilde{\mathbb{A}}^\mu)(\tilde{p}_\mu - e\tilde{\mathbb{A}}_\mu) &= g^{\nu\lambda}(\alpha)(\nabla_\nu(\alpha) - e\tilde{\mathbb{A}}_\nu(\alpha))(\nabla_\lambda(\alpha) - e\tilde{\mathbb{A}}_\lambda(\alpha)) \\ &= g^{\nu\lambda}(0)(\partial_\nu - e\mathbb{A}_\nu(0))(\partial_\lambda - e\mathbb{A}_\lambda(0)) \\ &= (p^\mu - e\mathbb{A}^\mu)(p_\mu - e\mathbb{A}_\mu). \end{aligned} \quad (5.18)$$

Superspinors are invariant with respect to the gauge transformations:

$$\tilde{\Psi}(x, \alpha) = e^{if(x)}\Psi(x, \alpha), \quad (5.19)$$

$$\tilde{\mathbb{A}}_\nu(x, \alpha) = \mathbb{A}_\nu(x, \alpha) - e^{-1}\varepsilon_\nu^\mu(\alpha)\partial_\mu f(x), \quad (5.20)$$

where $f(x)$ is an arbitrary real function of space-time coordinates.

6. SOLUTIONS

The modification of the Dirac equation effected by our prescription $\partial_\mu \Rightarrow \nabla_\mu$ leads to

$$(i\gamma^\mu \nabla_\mu - m)\Psi = 0, \quad (6.1)$$

$$(i\gamma^\mu \nabla_\mu - m)\Phi = 0, \quad (6.2)$$

corresponding to the ordinary positive and negative energy spinors:

$$(\gamma^\mu p_\mu - m)w = 0, \quad (6.3)$$

$$(\gamma^\mu p_\mu + m)u = 0. \quad (6.4)$$

We confine ourselves to a prototypical case - that of a boost in the x^3 direction. Specifically,

$$\nabla_0 = \varepsilon_0^0(\alpha)\partial_0 + \varepsilon_0^3(\alpha)\partial_3 + i\kappa_0(\alpha)K_3, \quad (6.5)$$

$$\nabla_3 = \varepsilon_3^0(\alpha)\partial_0 + \varepsilon_3^3(\alpha)\partial_3 + i\kappa_3(\alpha)K_3, \quad (6.6)$$

$$\nabla_1 = \partial_1, \quad (6.7)$$

$$\nabla_2 = \partial_2. \quad (6.8)$$

All other free superspinors can be obtained from these ones via the linear $SU_J(2)$ transformations (5.5). We look for plane-wave particle and antiparticle spinors ([5], Chapter XI, §§70-73) of the form

$$\Psi(\alpha) = w(\alpha)e^{-i(s_0(\alpha)x^0 + s_3(\alpha)x^3)}, \quad (6.9)$$

$$\bar{\Phi}(\alpha) = \bar{u}(\alpha)e^{-i(s_0(\alpha)x^0 + s_3(\alpha)x^3)}, \quad (6.10)$$

subject to the relativistic impulse condition $s_0^2(\alpha) - s_3^2(\alpha) = m^2$. This is a *conditio sine qua non* because every component $\Psi^l(\alpha)$ of $\Psi(\alpha)$ and $\bar{\Phi}^l(\alpha)$ must satisfy the Klein-Gordon equation

$$(\square + m^2)\Psi^l(\alpha) = 0, \quad (6.11)$$

$$(\square + m^2)\bar{\Phi}^l(\alpha) = 0. \quad (6.12)$$

In the standard representation

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{bmatrix}, \quad (6.13)$$

the equations (6.1) and (6.2) yield the following matrix:

$$\begin{bmatrix} \varepsilon_0(\alpha) - m(\alpha) & 0 & -\varepsilon_3(\alpha) - \kappa_0(\alpha) & 0 \\ 0 & \varepsilon_0(\alpha) - m(\alpha) & 0 & \varepsilon_3(\alpha) + \kappa_0(\alpha) \\ \varepsilon_3(\alpha) - \kappa_0(\alpha) & 0 & -\varepsilon_0(\alpha) - m(\alpha) & 0 \\ 0 & -\varepsilon_3(\alpha) + \kappa_0(\alpha) & 0 & -\varepsilon_0(\alpha) - m(\alpha) \end{bmatrix}, \quad (6.14)$$

where the entries are

$$\varepsilon_0(\alpha) = \varepsilon_0^0(\alpha)s_0 + \varepsilon_0^3(\alpha)s_3, \quad (6.15)$$

$$\varepsilon_3(\alpha) = \varepsilon_3^0(\alpha)s_0 + \varepsilon_3^3(\alpha)s_3, \quad (6.16)$$

$$m(\alpha) = m + \kappa_3(\alpha). \quad (6.17)$$

Its rank has to be 2 for all values of α , thus constraining $\kappa_0(\alpha)$ and $\kappa_3(\alpha)$:

$$\varepsilon_0^2(\alpha) - \varepsilon_3^2(\alpha) = (m + \kappa_3(\alpha))^2 - \kappa_0^2(\alpha). \quad (6.18)$$

At last, the proper role of $\kappa_\mu(\alpha)$'s is revealed: they make mass into a quantity that serves as such in noninertial frames. To keep this mass term position-invariant we must ensure that $\partial_\nu \kappa_\mu(\alpha) = 0$. The solutions are

$$w^{(1)}(\alpha) = \begin{bmatrix} \varepsilon_0(\alpha) + m + \kappa_3(\alpha) \\ 0 \\ \varepsilon_3(\alpha) - \kappa_0(\alpha) \\ 0 \end{bmatrix}, \quad w^{(2)}(\alpha) = \begin{bmatrix} 0 \\ \varepsilon_0(\alpha) + m + \kappa_3(\alpha) \\ 0 \\ -\varepsilon_3(\alpha) + \kappa_0(\alpha) \end{bmatrix}, \quad (6.19)$$

$$u^{(1)}(\alpha) = \begin{bmatrix} \varepsilon_3(\alpha) + \kappa_0(\alpha) \\ 0 \\ \varepsilon_0(\alpha) - m - \kappa_3(\alpha) \\ 0 \end{bmatrix}, \quad u^{(2)}(\alpha) = \begin{bmatrix} 0 \\ -\varepsilon_3(\alpha) - \kappa_0(\alpha) \\ 0 \\ \varepsilon_0(\alpha) - m - \kappa_3(\alpha) \end{bmatrix}. \quad (6.20)$$

The crucial values are $\alpha = \{0, \pi/2, \pi, 3\pi/2\}$. The first two:

$$\begin{aligned} \varepsilon_0^0(0) = \varepsilon_3^3(0) &= 1 & \varepsilon_0^0(\pi/2) = \varepsilon_3^3(\pi/2) &= \sqrt{2}/2 \\ \varepsilon_0^3(0) = \varepsilon_3^0(0) &= 0 & \varepsilon_0^3(\pi/2) = -\varepsilon_3^0(\pi/2) &= -\sqrt{2}/2 \\ \kappa_3(0) = \kappa_0(0) &= 0 & \lim_{\alpha \rightarrow \frac{\pi}{2}} ((m + \kappa_3(\alpha))^2 - \kappa_0^2(\alpha)) &= \infty \end{aligned}$$

mirror the second two:

$$\begin{aligned} \varepsilon_0^0(\pi) = \varepsilon_3^3(\pi) &= 0 \\ \varepsilon_0^3(\pi) = \varepsilon_3^0(\pi) &= -1 \\ (m - \kappa_3(\pi))^2 - \kappa_0^2(\pi) &= -m^2 \\ \varepsilon_0^0(3\pi/2) = \varepsilon_3^3(\pi/2) &= \sqrt{2}/2 \\ \varepsilon_0^3(3\pi/2) = -\varepsilon_3^0(3\pi/2) &= \sqrt{2}/2 \\ \lim_{\alpha \rightarrow \frac{3\pi}{2}} ((m + \kappa_3(\alpha))^2 - \kappa_0^2(\alpha)) &= -\infty. \end{aligned}$$

An unexpected relation between particles and antiparticles emerges:

$$\Psi^{(1)}(\alpha) = -\bar{\Phi}^{(1)}(\alpha + \pi), \quad (6.21)$$

$$\Psi^{(2)}(\alpha) = \bar{\Phi}^{(2)}(\alpha + \pi). \quad (6.22)$$

Verbally, virtualization and actualization in the context of space-time superspinor transformations occur only in conjunction with the charge conjugation. According to (6.21), (6.22), electrons are virtual positrons and *vice versa*. The superspinor formalism forestalls their leaving the mass surface, yet recognizes the difference between actual and virtual charged spin 1/2 particles associated with a frame. There is no question of moving with a superlight speed, for no particle would retain its original identity. In this brave new world electrons and positrons are just particular values of the superspinor wave function. Also, the particle-antiparticle symmetry hypothesis ([5], Chapter XI, §73) is ultimately vindicated, since the vacuum is filled with all kinds of negative energy superspinors, and must be electrically indefinite. Needless to say, the energy-impulse is commensurate with the frame rapidity, but the act of virtualization ($\Psi(\alpha) \mapsto \Psi(\alpha + \pi)$) must preserve it:

$$s_\mu(\alpha) = s_\mu(\alpha + \pi). \quad (6.23)$$

7. SUPERSPINOR STATISTICS

Even though the results of the previous section were obtained for a specialized UG -transformation, they obviously remain true for all superspinors. Thus α serves as a universal boost parameter.

An arbitrary solution allows plane-wave decompositions ([15], Chapter 4, §4.3):

$$\Psi(x) = \int \frac{d^3s}{(2\pi)^3} \frac{m}{s_0} \sum_{l=1,2} [\mathcal{B}_l(s) w^l(s) e^{-isx} + \mathcal{D}_l^H(s) u^l(s) e^{isx}], \quad (7.1)$$

$$\bar{\Psi}(x) = \int \frac{d^3s}{(2\pi)^3} \frac{m}{s_0} \sum_{l=1,2} [\mathcal{B}_l^H(s) \bar{w}^l(s) e^{isx} + \mathcal{D}_l(s) \bar{u}^l(s) e^{-isx}]. \quad (7.2)$$

In these formulas $\mathcal{B}_l(s)$ and $\mathcal{D}_l(s)$ are viewed as linear operators, not just coefficients, and would have to be interpreted as such. Flipping (7.1) and using (6.23), we get

$$\mathcal{B}_l(s(\alpha)) = (-1)^l \mathcal{D}_l^H(s(\alpha)). \quad (7.3)$$

It stands to reason, that, essentially, creating a particle is equivalent to annihilating an antiparticle.

By virtue of (7.3), superspinors entail the following anticommutators:

$$\{\mathcal{D}_l^H(s(\alpha)), \mathcal{B}_n(s'(\alpha))\} = \pm \{\mathcal{B}_l(s(\alpha)), \mathcal{B}_n(s'(\alpha))\} = \{\mathcal{B}_l(s(\alpha)), \mathcal{D}_n^H(s'(\alpha))\}, \quad (7.4)$$

$$\{\mathcal{B}_l^H(s(\alpha)), \mathcal{D}_n(s'(\alpha))\} = \pm \{\mathcal{D}_l(s(\alpha)), \mathcal{D}_n(s'(\alpha))\} = \{\mathcal{D}_l(s(\alpha)), \mathcal{B}_n^H(s'(\alpha))\}. \quad (7.5)$$

Whenever $s(\alpha) \neq s'(\alpha)$, the only way for the left- and right-hand side anticommutators to be equal is to vanish, because creating a particle with impulse s combined with annihilating an antiparticle with impulse s' is fundamentally different from creating a particle with impulse s' combined with annihilating an antiparticle with impulse s . Now the above anticommutators must continuously depend on the impulse. Therefore

$$\{\mathcal{B}_l(s(\alpha)), \mathcal{B}_n(s'(\alpha))\} = \{\mathcal{D}_l(s(\alpha)), \mathcal{D}_n(s'(\alpha))\} = 0, \quad \forall s, s'. \quad (7.6)$$

Hence $\mathcal{B}_l(s(\alpha))\mathcal{B}_l(s(\alpha)) = 0$, and furthermore $\mathcal{B}_l(s(\alpha))\mathcal{B}_l(s(\alpha))| \rangle = 0$. This in fact says that two superspinors with the definite impulse s , an identical spin, and an identical charge cannot be in the same state. We conclude that for superspinors, the

Fermi-Dirac statistics comes about as a direct consequence of the relativistic covariance, whereas the conventional Dirac spinors need additional anticommutator relations - the Jordan-Wigner postulates ([15], Chapter 4, §4.3).

8. TWIN PARADOX FOR SUPERSPINORS

A simple way to determine whether the superspinor model has any semblance to the real world is to conduct an experiment in the setting similar to that of the twin paradox experiment. Let us let one local frame move, while the other be still. Let there be an electromagnetic field expressible in the moving frame as $-e\mathbb{A}_\mu$. At the exact moment these two frames coincide in space, let that exact location be bombarded with a gravitational wave decomposable into two pieces: $\Gamma = (\kappa_\mu^a K_a + \Re(e\mathbb{A}_\mu))$. Then $\pm\kappa_\mu^a K_a$ and $\pm\Re(e\mathbb{A}_\mu)$ cancel, and electrons (in fact, any massive spin 1/2 particles participating in the electromagnetic interactions) would behave differently in these two frames. Now apply a uniform gravitational wave over a region in space. Let the moving frame be associated with a spacecraft. When it finally returns to the location of the resting local frame, the differences in the electron superspinors congeal and become absolute. More specifically, the electrons on the spaceship would be impervious to the action of the uniform gravitational wave - a levitation of sorts.

This can be seen as a mirror image of the Aharonov-Bohm [1] phenomenon. Indeed, a change in the fermion field triggered on the moving spaceship by the uniform gravitational wave (which essentially is a space-time deformation, albeit not necessarily topologically nontrivial one) causes changes in the electromagnetic field. Globally, there is an interdependency between massive spin 1/2 particles and electromagnetic fields. Direct interaction cannot account for all of that interdependency. We would like to call it the Aharonov-Bohm symmetry. Its secret is hidden deep in the topology and small-scale structure of the space-time. We can only speculate that such conundrums as the self-action of the electric field of an electron, or the electromagnetic mass will find some measure of elucidation within a framework encompassing the Aharonov-Bohm symmetry.

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